Algebraic and topological regularity properties of nuclear C^*-algebras

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Nuclear C*-algebras

Algebraic and topological regularity

Topological dimension

The Jiang–Su algebra $\mathcal{Z}$

The regularity conjecture

DEFINITION

A C*-algebra is a Banach $*$-algebra A satisfying

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$.

(Equivalently, a C*-algebra is a norm-closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.)

We define the cone of positive elements of A by

$$A_+ := \{a^*a \mid a \in A\}.$$

A map

$$\varphi : A \to B$$

is a $*$-homomorphism, if it is linear, multiplicative, and $*$-preserving.
φ is completely positive, if it is linear, $*$-preserving, and

$$\varphi^{(n)} : M_n(A) \to M_n(B)$$

is positive for all $n \in \mathbb{N}$.

EXAMPLES

- $\mathcal{C}_0(X)$ for X locally compact
- $\mathcal{C}_0(X) \otimes M_r$
- sums, pullbacks and inductive limits of the above
  (AH and ASH algebras)
- for X compact and $\beta : X \to X$ a homeomorphism,

$$\mathcal{C}(X) \rtimes \mathbb{Z} := C^*(\mathcal{C}(X), u \mid u \text{ a unitary with } uf(\,.\,)u^* = f(\beta(\,.\,)))$$

THEOREM (Choi–Effros)
A is nuclear iff there is a system

$$(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A)_\Lambda$$

of (finite dimensional) c.p.c. approximations for A such that

$$\varphi_\lambda \psi_\lambda \to \mathrm{id}_A$$

pointwise.

We will think of such c.p.c. approximations as noncommutative partitions of unity.

CONJECTURE (Elliott)
Separable nuclear C*-algebras are classified by K-theoretic data.

K-theory is a (computable) homological invariant based on equivalence classes of projections and of unitaries.

But why *nuclear* C*-algebras?

REMARKS

► Nuclearity is a flexible concept; it can be characterized in many different ways, which make contact with many areas of operator algebras.

► Finite-dimensional approximations seem promising, but c.p. approximations are not a natural framework to study K-theoretic data.

DEFINITION (Kadison–Kastler)

Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C*-algebras acting on the same Hilbert space. We write $d(A, B) < \gamma$, if the unit balls of A and B are within γ of each other.

There is also a one-sided version (due to Christensen).

CONJECTURE (Kadison–Kastler)

If $A, B \subset \mathcal{B}(\mathcal{H})$ are separable C*-algebras and $d(A, B) < \gamma$ for some small enough γ, then A and B are (unitarily) isomorphic.

THEOREM (Christensen–Sinclair–Smith–White–W, 2009)
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C*-algebras, with A separable and nuclear and $d(A, B) < 10^{-6}$.
Then, $A \cong B$.

THEOREM (Christensen–Sinclair–Smith–White–W, 2009)
For $n \in \mathbb{N}$ there is $\gamma > 0$ such that the following holds:
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C*-algebras, with A separable and $\dim_{\mathrm{nuc}} A \leq n$, and with $A \subset_\gamma B$.
Then, there exists an embedding $A \hookrightarrow B$.

REMARK The first result shows that nuclear C*-algebras are stable under small perturbations; the same holds for K-theoretic invariants.

The second result has recently been generalized (using very similar methods together with a result of Kirchberg) to *all* separable nuclear C*-algebras (Hirshberg–Kirchberg–White, 2011).

Consider the following regularity properties for a C*-algebra A.

(A) A is topologically finite-dimensional.

(B) A absorbs a suitable strongly self-absorbing C*-algebra tensorially.

(Γ) A allows comparison of its positive elements in the sense of Murray and von Neumann.

What do these properties mean?
How are they related?
What can they do for us?

They may be interpreted as topological, (C*-)algebraic, and homological regularity properties, respectively.

DEFINITION

A c.p.c. map $\varphi : A \to B$ has order zero, if it respects orthogonality, i.e.

$$(e \perp f \in A_+ \Rightarrow \varphi(e) \perp \varphi(f) \in B_+).$$

DEFINITION (Zacharias–W; Kirchberg–W)
Let A be a C*-algebra, $n \in \mathbb{N}$. We say A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\epsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with F finite dimensional, ψ c.p.c., and

$$\varphi \circ \psi =_{\mathcal{F}, \epsilon} \mathrm{id}_A,$$

and such that F can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with c.p.c. order zero maps

$$\varphi^{(i)} := \varphi|_{F^{(i)}}.$$

We say A has decomposition rank at most n, $\mathrm{dr}\, A \leq n$, if in addition the map $\sum_i \varphi^{(i)}$ can be chosen to be contractive.

DEFINITION

A has locally finite nuclear dimension (or decomposition rank), if for any finite $\mathcal{F} \subset A$ and $\epsilon > 0$ there is $B \subset A$ such that $\dim_{\mathrm{nuc}} B < \infty$ (or $\mathrm{dr}\, B < \infty$) and $\mathcal{F} \subset_{\epsilon} B$.

REMARKS

- We have $\dim_{\mathrm{nuc}} A \leq \mathrm{dr}\, A \leq \dim_{\mathrm{ASH}} A \leq \dim_{\mathrm{AH}} A$.
- Locally finite nuclear dimension implies nuclearity.
- We do not know of any nuclear C^*-algebra which does not have locally finite nuclear dimension.

The Jiang–Su algebra $\mathcal{Z}$

- is the uniquely determined initial object in the category of strongly self-absorbing C^*-algebras (W, 2009).

- can be written as a universal C^*-algebra with countably many generators and relations based on

$$
Z^{(p)} := C^* \left(\varphi^{(p)}, \psi^{(p)} \;\middle|\; \begin{array}{l} \varphi^{(p)} \text{ is a c.p.c. order zero map on } M_p, \\ \psi^{(p)} \text{ is a c.p.c. order zero map on } M_2, \\ \psi^{(p)}(e_{22}) = \mathbf{1} - \varphi^{(p)}(\mathbf{1}_p), \\ \varphi^{(p)}(e_{11}^{(p)})\psi^{(p)}(e_{11}^{(2)}) = \psi^{(p)}(e_{11}^{(2)}) \end{array} \right)
$$

and (explicit) relations between $Z^{(p)}$ and $Z^{(p^3)}$ (Jacelon–W, 2011).

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CONJECTURE (Toms–W, 2007)

For a nuclear, separable, simple, unital and nonelementary C*-algebra A, t.f.a.e.:

(i) A has finite nuclear dimension

(ii) A is $\mathcal{Z}$-stable (i.e., $A \cong A \otimes \mathcal{Z}$)

(iii) A has strict comparison of positive elements

(i.e., whenever $a, b \in A_+$ satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$, then $a \precsim b$).

REMARKS

► The conjecture can also be formulated in the nonunital and nonsimple case.

► In the finite case, replace 'nuclear dimension' by 'decomposition rank' in (i).

► (iii) may as well be interpreted as order-completeness of a homological invariant.

► (i) $\implies$ (ii) was shown by W (2008/10), (in the finite and in the general case).

► (ii) $\implies$ (iii) was shown by Rørdam.

► (ii) $\implies$ (i) is known in many cases, using classification results.

► (iii) $\implies$ (ii) holds in fair generality, see below.

DEFINITION

We say a separable, simple, unital, nuclear C^*-algebra A is pure if

- it has strict comparison

- it is almost divisible

  (i.e., for any $a \in A_+$ and $k \in \mathbb{N}$ there is $b \in A_+$ with $k \cdot b \precsim a \precsim (k+1) \cdot b$).

THEOREM (W, 2010)
Let A be separable, simple, unital, with locally finite nuclear dimension.
If A is pure, then A is $\mathcal{Z}$-stable.

This is (iii) $\Longrightarrow$ (ii) of the conjecture above, at least in the case of
locally finite nuclear dimension and almost divisibility.

THEOREM (Toms, 2010)
Simple, unital AH algebras with slow dimension growth are pure.

COROLLARY (using results of Lin, Lin–Niu, W)
Simple, unital AH algebras with slow dimension growth satisfy the
Elliott conjecture.

(This generalizes Elliott–Gong–Li classification of simple unital AH
algebras with very slow dimension growth.)

QUESTION
Is a simple, unital C^*-algebra with strict comparison almost divisible?

Ingredients of the proof (that locally finite nuclear dimension + pure implies $\mathcal{Z}$-stable):

By Toms–W, we have to produce an approximately central sequence of unital $*$-homomorphisms

$$Z_{p,p+1} \to A$$

for any $p \in \mathbb{N}$, where

$$Z_{p,p+1} = \{f \in \mathcal{C}([0,1], M_p \otimes M_{p+1}) \mid f(0) \in M_p \otimes 1_{M_{p+1}}, f(1) \in 1_{M_p} \otimes M_{p+1}\}.$$

By Rørdam–W, we need to find a c.p.c. order zero map

$$\Phi : M_p \to A$$

and $v \in A$ such that

$$vv^* = \mathbf{1}_A - \Phi(\mathbf{1}_{M_p}) \text{ and } v^*v \leq \Phi(e_{11})$$

and such that $\Phi(M_p)$ and v are approximately central.

The following is the key result for constructing both Φ and v.

LEMMA

For $m \in \mathbb{N}$, there is $\alpha_m > 0$ such that the following holds:

Let A be separable, simple, unital and pure.

Let $\mathbf{1}_A \in B \subset A$ be a C*-subalgebra with $\dim_{\mathrm{nuc}} B \leq m$, and let $k, l \in \mathbb{N}$. If

$$\varphi : M_l \to A_\infty \cap B'$$

is c.p.c. order zero, then there is a c.p.c. order zero map

$$\Psi : M_k \to A_\infty \cap B' \cap \varphi(M_l)'$$

such that

$$\tau(\Psi(\mathbf{1}_k)\varphi(\mathbf{1}_l)b) \geq \alpha_m \cdot \tau(\varphi(\mathbf{1}_l)b)$$

for all $b \in B_+$ and $\tau \in T_\infty(A)$.